Activated random walk

Jacob Richey

joint with: Riddhi Basu, Shirshendu Ganguly, Chris Hoffman, Leo Rolla

University of Washington

jfrichey@uw.edu

CUNY, 4/28/2020

ARW basics

ARW (activated random walk) is a random particle processes on a graph.
Relatives: SM & SSM (abelian/stochastic sandpiles).

The input for ARW is:

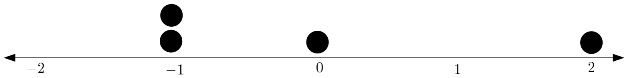
- G a (connected) graph (think $\mathbb{Z}^d$)
- $\mu \in [0, \infty)$ the 'mass' parameter
- $\lambda \in (0, \infty]$ the 'sleep rate'

ARW basics

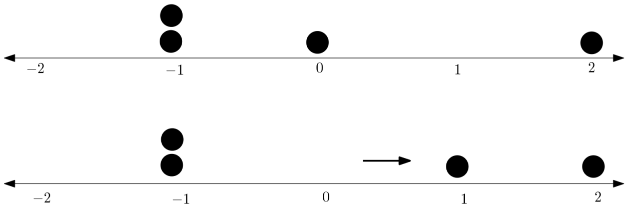
Initially, each vertex of G has an iid random number of particles with mean μ.

- The particles can be in two states: active and asleep. All particles are initially active.
- Active particles perform independent simple random walks at rate 1
- Active particles fall asleep at rate λ
- Sleepy particles do not move
- When two or more particles occupy the same site, all particles at that site become active

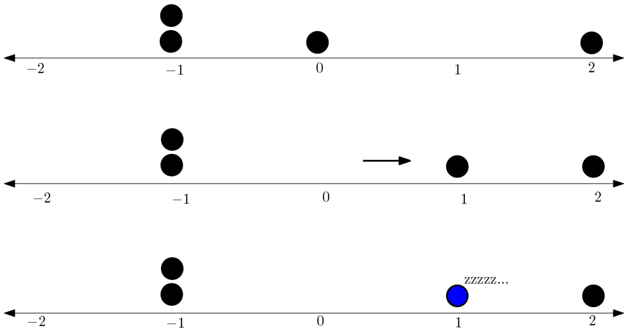
ARW on $\mathbb{Z}$



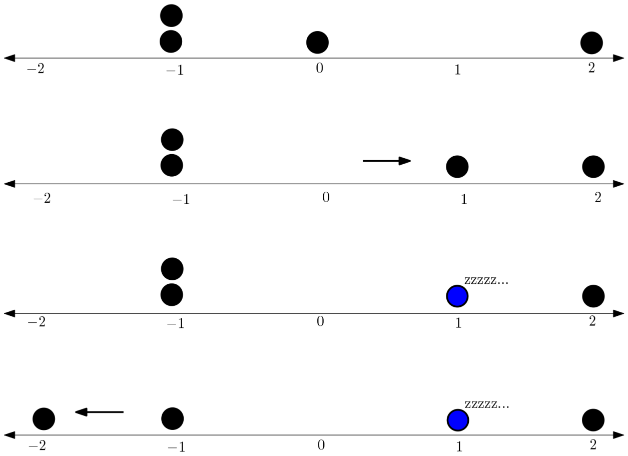
ARW on $\mathbb{Z}$



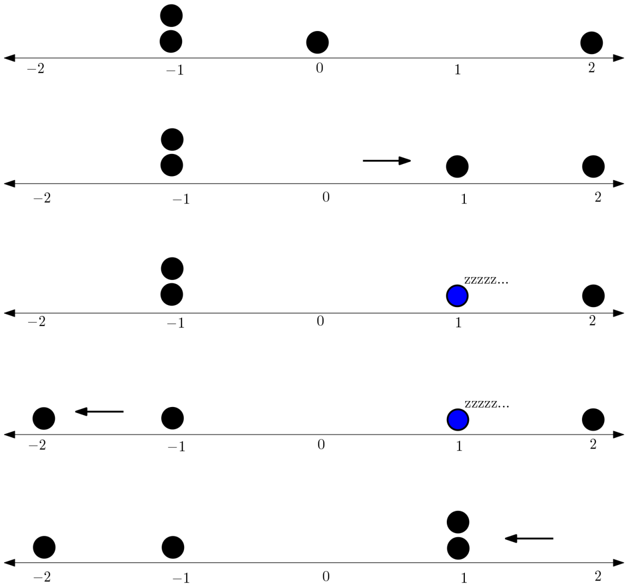
ARW on $\mathbb{Z}$



ARW on $\mathbb{Z}$



ARW on $\mathbb{Z}$



Self organized criticality

A dynamical system exhibits ‘self organized criticality’ (SOC) if the system evolves to a critical state without any fine-tuning of model parameters ($+$ power law avalanches)

Forest fires, chemical reactions, electrical networks, ...

Prototypical deterministic model of SOC: Abelian sandpile

Self-organized criticality

'Driven-dissipative' version of ARW:

1. Start with any(!) initial configuration of particles on $[-N, N]^d \subset \mathbb{Z}^d$
2. Run ARW dynamics, killing particles that hit the boundary of the box
3. When all particles are asleep, add a particle at a random site
4. Return to step 2

Different from ARW on a graph: no mass conservation!

Self-organized criticality

- Existence of critical density?
- How quickly does the density converge? Fluctuations?
- How can the critical distribution be characterized/described?
- How long does the system take to stabilize above/below criticality?
- How ‘strong’ is the push towards the critical point?

Self-organized criticality

- Existence of critical density?
- How quickly does the density converge? Fluctuations?
- How can the critical distribution be characterized/described?
- How long does the system take to stabilize above/below criticality?
- How 'strong' is the push towards the critical point?

Conjecturally:

$$\text{SOC for finite driven-dissipative system}$$
$$\Longleftrightarrow$$
$$\text{Absorbing state phase transition for infinite system}$$

Fixation

Local fixation

An instance of ARW is said to *fixate* if each vertex in G is visited a finite number of times by active particles.

Fixation

Local fixation

An instance of ARW is said to *fixate* if each vertex in G is visited a finite number of times by active particles.

What is the probability of fixation? How does it depend on μ and λ?

If G is finite, ARW eventually fixates. How fast does fixation occur?

Critical density

Let $G = \mathbb{Z}^d$. We are interested in $\mathbb{P}(\text{ARW}(\mu, \lambda) \text{ fixates})$.

Ergodicity

For any particle density μ and sleep rate λ, $\mathbb{P}(\text{ARW}(\mu, \lambda) \text{ fixates}) \in \{0, 1\}$.

Monotonicity

If $\text{ARW}(\mu_0, \lambda_0)$ fixates almost surely for some μ_0 and λ_0, then it fixates almost surely for any $\mu \leq \mu_0$.

Critical density

Monotonicity $+$ 0-1 law $\implies$ critical density μ_c

Existence of the critical density

For each $\lambda \in [0, \infty]$, there is a critical density $\mu_c(\lambda) \in [0, 1]$ satisfying

$$\mathbb{P}(ARW(\mu, \lambda) \text{ fixates}) = \begin{cases} 1, \mu < \mu_c \\ 0, \mu > \mu_c \end{cases}$$

Diaconis-Fulton representation

A useful equivalent construction in discrete time:

Define iid instructions $\{\xi_{v,j}\}$ for $v \in G$ and $j \geq 1$ by

$$\xi_{v,j} = \begin{cases} \text{move the particle at } v \text{ to a uniform neighbor} \\ \text{of } v \text{ with probability } \frac{1}{\deg(v)(1+\lambda)} \\ \text{put the particle at site } v \text{ to sleep with probability } \frac{\lambda}{1+\lambda} \end{cases}$$

At each site v, the jth time we topple a particle at v, the state of the system changes according to $\xi_{v,j}$.

Diaconis-Fulton representation

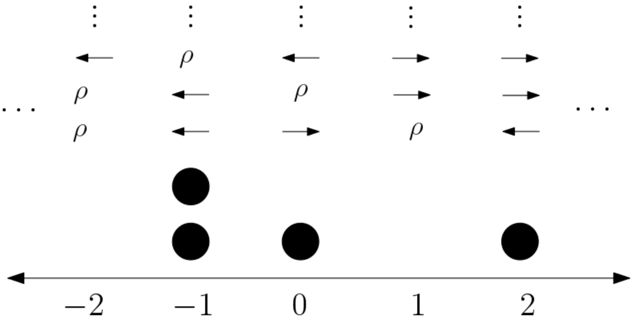


Figure: The stacks of instructions, visualized as lying above each vertex of the graph $G = \mathbb{Z}$. ρ represents a 'sleep' instruction.

Issue: what if the evolution of the system depends on the order of the instructions?

Diaconis-Fulton representation

The order doesn't matter!

Abelian property

Consider any initial configuration η, and any (legal) sequence of stack instructions $\overline{\xi} = (\xi_{v^1,j^1}, \xi_{v^2,j^2}, \ldots, \xi_{v^N,j^N})$. If $\overline{\xi}'$ is any (legal) re-ordering of the instructions in $\overline{\xi}$, then $\overline{\xi}\eta = \overline{\xi}'\eta$.

This is a useful framework for proving rigorous results: we can choose clever toppling sequences.

Results

Consider ARW on $\mathbb{Z}$.

Rolla, Sidoravicius '12

$\mu_c \in [\frac{\lambda}{1+\lambda}, 1]$.

Basu, Ganguly, Hoffman '15

For any $\mu > 0$, there exists $\lambda_\mu > 0$ such that $ARW(\mu, \lambda)$ does not fixate for $\lambda < \lambda_\mu$.

Hoffman, R., Rolla '20

For any $\lambda > 0$, $\mu_c(\lambda) < 1$.

Results

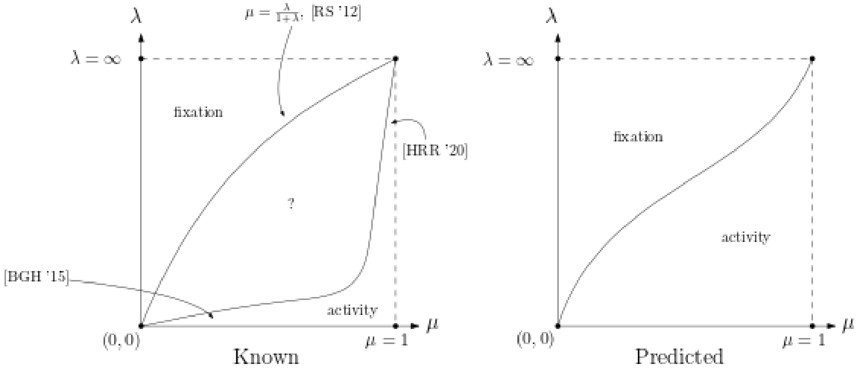


Figure: Phase diagram for ARW on $\mathbb{Z}$.

Results

Let $T(\mu, \lambda)$ denote the number of stack instructions needed to fixate ARW on $G = \mathbb{Z}/n\mathbb{Z}$.

Basu, Ganguly, Hoffman, R. '17

- For any $\lambda \in (0, \infty]$ and $\mu < \frac{\lambda}{1+\lambda}$,

$$T(\mu, \lambda) < Cn \log(n)^2$$

  with high probability as $n \to \infty$ for some $C > 0$.
- For any $\mu \in (0, 1)$, there exists $\lambda_0 > 0$ such that for $\lambda < \lambda_0$,

$$T(\mu, \lambda) > e^{cn}$$

  with high probability as $n \to \infty$ for some $c > 0$.

Proof of fixation

Rolla, Sidoravicius '12

ARW fixates almost surely on $G = \mathbb{Z}$ for $\mu < \frac{\lambda}{\lambda+1}$.

Proof sketch: find 'traps' for the particles to fall asleep in.

Proof of fixation

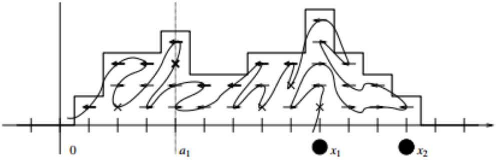


Figure: A diagram from [RS '12], showing the first trap a_1 for the particle x_1.

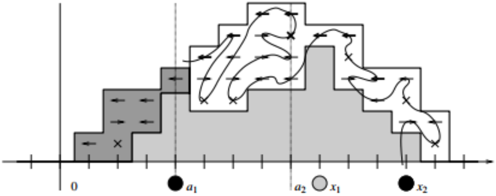


Figure: The trap a_2 for the particle x_2, obtained recursively by exploring the stack instructions.

Proof of fixation

Let x_k = position of kth particle to the right of 0, $k = 1, 2, \ldots$. Define the traps a_k recursively:

- $a_0 = 0$.
- For $k > 0$: send a ghost particle out from x_k, ignoring sleep instructions, until it hits a_{k-1}.
- a_k = leftmost site to the right of a_{k-1} where the second to last instruction seen by the ghost was a sleep instruction.

Particles follow the paths of their ghosts, except that they fall asleep in the trap.

Proof of fixation

Proof sketch: Trap setting succeeds if $a_{k-1} < x_k$ for all $k \in \mathbb{N}$.

Proof of fixation

Proof sketch: Trap setting succeeds if $a_{k-1} < x_k$ for all $k \in \mathbb{N}$.

On average, $x_k - x_{k-1} = \mu^{-1}$ and $a_k - a_{k-1} = \frac{1+\lambda}{\lambda}$.

Proof of fixation

Proof sketch: Trap setting succeeds if $a_{k-1} < x_k$ for all $k \in \mathbb{N}$.

On average, $x_k - x_{k-1} = \mu^{-1}$ and $a_k - a_{k-1} = \frac{1+\lambda}{\lambda}$.

By the LLN, $x_k \approx k\mu^{-1}$, and $a_k \approx k \cdot \frac{1+\lambda}{\lambda}$.

Proof of fixation

Proof sketch: Trap setting succeeds if $a_{k-1} < x_k$ for all $k \in \mathbb{N}$.

On average, $x_k - x_{k-1} = \mu^{-1}$ and $a_k - a_{k-1} = \frac{1+\lambda}{\lambda}$.

By the LLN, $x_k \approx k\mu^{-1}$, and $a_k \approx k \cdot \frac{1+\lambda}{\lambda}$.

Thus if $\mu < \frac{\lambda}{1+\lambda}$, $x_k > a_k$ for all k large a.s.

Proof of fixation

Proof sketch: Trap setting succeeds if $a_{k-1} < x_k$ for all $k \in \mathbb{N}$.

On average, $x_k - x_{k-1} = \mu^{-1}$ and $a_k - a_{k-1} = \frac{1+\lambda}{\lambda}$.

By the LLN, $x_k \approx k\mu^{-1}$, and $a_k \approx k \cdot \frac{1+\lambda}{\lambda}$.

Thus if $\mu < \frac{\lambda}{1+\lambda}$, $x_k > a_k$ for all k large a.s.

So $\mathbb{P}(\text{fixation}) > 0$. By the 0-1 law, $\mathbb{P}(\text{fixation}) = 1$. $\square$

Fast fixation on the cycle

Basu, Ganguly, Hoffman, R. '17

Consider *ARW* on $\mathbb{Z}/n\mathbb{Z}$. For any $\lambda \in (0, \infty]$ and $\mu < \frac{\lambda}{1+\lambda}$,

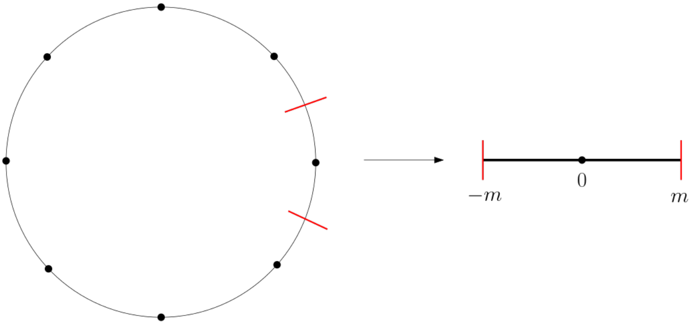
$$T(\mu, \lambda) < Cn \log(n)^2$$

with high probability as $n \to \infty$ for some $C > 0$.

The fixation speed depends on the initial condition: if all particles start at the same site, $T \geq O(n^3)$ whp.

Fast fixation on the cycle

First step: gather $O(\log n)$ particles at each of $O\left(\frac{n}{\log n}\right)$ sites.



Focus on a single sub-interval.

How to adapt the traps for an interval?

We use a two-sided version of the trap setting on a finite interval, where the traps start at the boundary and move toward 0.

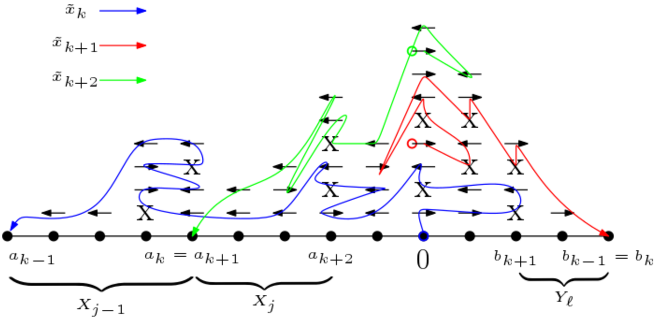


Figure: Setting traps 'in both directions' on an interval.

Fast fixation on the cycle

Internal erosion on an interval:

1. Start with the interval $X_0 = [-m, m] \cap \mathbb{Z}$.
2. Start a simple random walker from 0, stopped when she hits a boundary point $B \in \partial X_t$.
3. Remove the point B from X_0, to obtain $X_{t+1} = X_t \setminus \{B\}$.
4. Return to step 2

Fast fixation on a cycle

Idea: replace each segment $[j-1,j]$ and $[-j,-j+1]$ by an independent Exponential(j) length of rope, connect them all together, and initialize by lighting both ends on fire.

Properties of exponentials give a coupling between this process and the erosion process. Key computation: for $a,b>0$,

$$\mathbb{P}^0(\text{hit } b \text{ before } -a) = \frac{a}{a+b} = \mathbb{P}(\text{Exp}(b) < \text{Exp}(a)).$$

($+$ memoryless-ness)

Fast fixation on the cycle

Levine, Peres, '07

Let $R(m)$ be the number of sites remaining when the origin is eroded. As $m \to \infty$,

$$\frac{R(m)}{m^{3/4}} \to_d \left(\frac{8}{3}\right)^{1/4} \sqrt{|Z|},$$

where $Z \sim N(0,1)$.

Note: the number of remaining sites is $O(m^{3/4}) = o(m)$.

Fast fixation on the cycle

Issue: at each stage, one of the traps moves a random distance – distributed as $\text{Geo}\left(\frac{1+\lambda}{\lambda}\right)$ – not distance 1.

We are still able to couple with the rope process, but the exponentials have random means. Many concentration estimates necessary.

Conclusion: the left and right side traps still shrink to 0 at the same rate (up to lower order stuff). Two-sided trap setting succeeds for $\mu < \frac{\lambda}{1+\lambda}$.

Proof of non-fixation

Hoffman, R., Rolla '20

For any $\lambda > 0$, $\mu_c(\lambda) < 1$.

Main tool:

Rolla '19

Consider ARW on the interval $D_r = (0, r)$, and let M_r be the number of particles that exit D_r after stabilizing all sites in D_r. If

$$\limsup_{r \to \infty} \frac{\mathbb{E} M_r}{r} > 0,$$

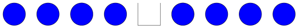
then ARW does not fixate a.s.

Proof of non-fixation

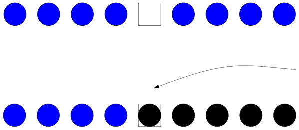
Start with a 'carpet' configuration: most sites have 1 particle, with regularly spaced 'holes' (with 0 particles).

Focus on a single block of carpet, and count how many particles are emitted to neighboring blocks.

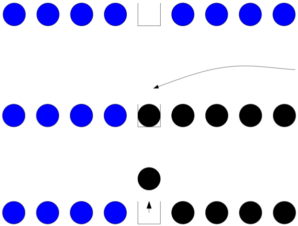
Proof of non-fixation



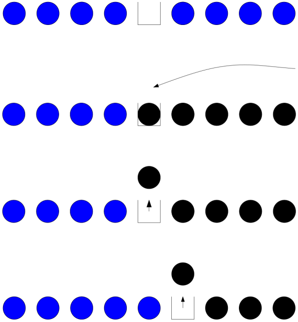
Proof of non-fixation



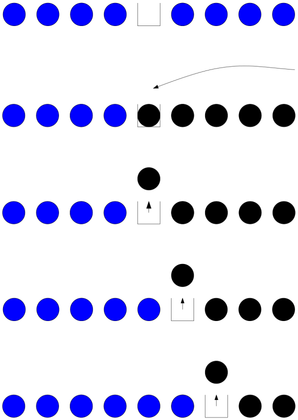
Proof of non-fixation



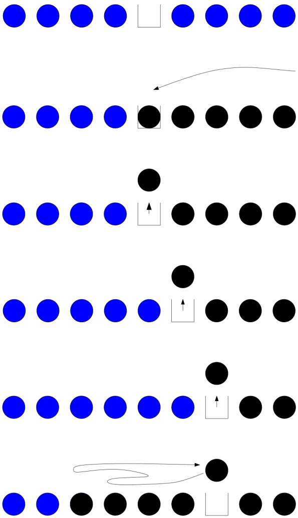
Proof of non-fixation



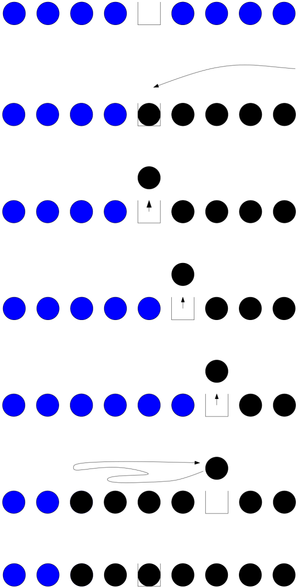
Proof of non-fixation



Proof of non-fixation



Proof of non-fixation



Proof of non-fixation

Activate one particle at a time. Always activate the closest active particle to the center of the block.

The hole acts like a random walk, with a strong bias towards 0. (Expected maximum distance reached by a SRW excursion is infinite!)

Show that particles are emitted from the interval much more often than the hole moves far away from the center.

Conclusion: many particles exit the interval before our scheme gets disrupted!

To do list

- Extend methods to work for the stochastic sandpile.
- For the ring: sharpen the phase transition. Behavior near criticality?
- Understand the odometer function. For example, starting from n particles at 0 (oil & water).

References

- *Activated random walks on $\mathbb{Z}$*. Leonardo Rolla. 2019, preprint (arXiv:1906.05037)
- *Non-fixation for ARW on $\mathbb{Z}$ for any sleep rate.* Chris Hoffman, Jacob Richey, Leonardo Rolla. (In preparation; 2020)
- *Activated random walk on a cycle*. Riddhipratim Basu, Shirshendu Ganguly, Chris Hoffman, Jacob Richey. Published in *AIHP*, 2019. (arXiv:1709.09163)
- *Absorbing-state phase transition for driven-dissipative stochastic dynamics on $\mathbb{Z}$*. Leonardo T. Rolla and Vladas Sidoravicius. Inventiones mathematicae, 188(1):127–150, 2012.
- *Non-fixation for Conservative Stochastic Dynamics on the Line*. Riddhipratim Basu, Shirshendu Ganguly, and Christopher Hoffman. Communications in Mathematical Physics 358(6), December 2017